LETTER TO THE EDITOR

The critical behaviour of clusters with no free ends

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Abstract. The number of clusters with no free ends $C(n)$ is enumerated up to 21 bonds on a square lattice for bond dilution. The exponent ν governing the growth of the radius of gyration of the cluster is also calculated. It is found that $C(n) \sim n^{-0.56} 3.15^n$ and $\nu = 0.54 \pm 0.01$.

The self-avoiding polygon has been studied extensively [1–5]. It is found that the exponent of its radius of gyration ν is the same as that of the end-to-end distance of the self-avoiding walk. Because the self-avoiding polygon is just a one-loop diagram, it would be interesting to study the exponent of the radius of gyration for diagrams with more loops, i.e. the diagrams with no free ends (NFE). Another motivation for enumerating the diagrams with NFE is that these diagrams are very useful in series expansions for a number of problems such as lattice animals, localisation and percolation. Harris [6] has formulated an effective single-site potential which eliminates free ends from the diagrammatic series expansion. Hence, for classical problems, an enumeration of diagrams having NFE is sufficient for the low-concentration series and one can thereby get much longer series. The effect of the loops has been studied [7, 8] for lattice animals. An important conclusion drawn from the work of Lubensky and Isaacson [9] is that the loops are irrelevant for the lattice animals. Series analyses [7, 8] of the lattice animals with a prescribed number of cycles confirmed the conclusion that ν_C is independent of C, where C is the number of the cycle in the animals and ν_C is the corresponding correlation length exponent.

In this letter, we consider the effect of adding more loops to the one-loop diagram. We have enumerated the number of the diagrams with no free ends and calculated the exponent of the radius of gyration on a square lattice for bond dilution. We found that the exponent of the radius of gyration for the diagram with no free ends $\nu_{NFE} = 0.54$ which is different from that for the one-loop diagram $\nu = 3/4$ [10] and from that of the lattice animal $\nu = 0.64$ [11].

We assume that the number of diagrams with no free ends having n bonds $C(n)$ obeys the following scaling form:

$$C(n) \sim n^{-\theta} \lambda^n \tag{1}$$

where θ is an exponent. Because we are considering bond dilution, the square radius of gyration should be defined in terms of bond coordinates. The coordinate of bond

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i is defined as that of the midpoint of the bond. Thus the square radius of gyration with respect to the centre of mass is defined as

$$R_n^2 = \frac{1}{n} \sum_i |r_i - r_c|^2 \tag{2}$$

and the mean-square radius of gyration $\rho(n)$ is given by

$$\rho(n) = \sum_{\gamma_n} \frac{R_n^2(\gamma_n)}{C(n)} \sim n^{2\nu_{\text{NFE}}} \tag{3}$$

where r_c is the centre of mass and γ_n denotes all of the diagrams having n bonds. Here we set $\rho(n) = 0$ if $C(n) = 0$.

We have calculated $C(n)$ and $\rho(n)$ on a square lattice up to 21 bonds using a partial enumeration method [12] based on the backtracking algorithm [13]. This calculation was done on an Apollo AD 4500 work station (a little slower than a VAX 8650) and took about 99 hours of CPU time. The coefficients $C(n)$ and $nC(n)\rho(n)$ are listed in table 1. We have analysed the series using the Padé approximant and the differential Padé approximant [14]. We first analysed the series χ_1 defined as

$$\chi_1 = \sum_n n\rho(n)K^n \sim |K - 1|^{-2\nu_{\text{NFE}}-2} \tag{4}$$

where K is the fugacity. Since the pole is exactly one, we can get ν_{NFE} with greater accuracy. We then analysed χ_2 and χ_3 defined as

$$\chi_2 = \sum_n C(n)K^n \sim |K\lambda - 1|^{\theta-1} \tag{5}$$

$$\chi_3 = \sum_n nC(n)\rho(n)K^n \sim |K\lambda - 1|^{\theta-2\nu_{\text{NFE}}-2} \tag{6}$$

Table 1. The coefficients of the series.

n	$C(n)$	$nC(n)\rho(n)$
1	0	0.0
2	0	0.0
3	0	0.0
4	1	1.0
5	0	0.0
6	2	8.0
7	2	8.0
8	9	73.0
9	20	192.9
10	68	1018.2
11	174	3 358.6
12	577	14 915.0
13	1 586	51 669.6
14	5 088	207 616.5
15	14 604	726 262.7
16	45 899	2 745 435.0
17	135 456	9 527 305.0
18	420 640	33 970 370.0
19	1 263 060	105 794 800.0
20	3 902 262	266 624 100.0
21	11 842 828	452 469 800.0

and used the value ν_{NFE} that we got from χ_1 to determine λ and θ. We obtained $\nu_{NFE} = 0.54 \pm 0.01$, $\theta = 0.56 \pm 0.01$, and $\lambda = 3.15 \pm 0.11$. Note that for the one-loop polygon $\nu = 3/4$ [10] and for lattice animals $\nu = 0.64$ [11]. Because the diagrams having NFE are more compact, ν_{NFE} should be smaller than that of the one-loop diagrams and that of the lattice animals.

In summary, we have enumerated the number of the diagrams having NFE and its exponent of the radius of gyration. We found that $\nu_{NFE} = 0.54$ which is different from that of the one-loop diagrams and that of the lattice animals.

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